Dissipative corrections to escape probabilities of thermally nonequilibrium systems

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Dissipative corrections to the *logarithm* of the probability of escape from a metastable state of an underdamped Markov system are considered. The approach proposed is based on the Poincaré cross-section method applied to the optimal path along which a system moves, with an overwhelming probability, in the course of a fluctuation resulting in escape. The corrections depend crucially on the structure of the trajectories of motion of the system near the saddle point in the absence of dissipation and fluctuations. For two important patterns of the trajectories the corrections are $\sim \eta^2 \ln\eta$ and $\sim \eta$ where η is a dissipation parameter. Numerical analysis of the escape probabilities for a nonlinear oscillator bistable in a nearly resonant field is fulfilled and the results are shown to be in good agreement with the analytical predictions. A feature of the pattern of the optimal paths in systems without detailed balance, the onset of caustics, is revealed.

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I. INTRODUCTION

The problem of the influence of dissipation on the probability of escape from a metastable state has been extensively studied in recent years, primarily in the context of quantum tunneling with dissipation, in particular that in Josephson junctions and superconducting quantum interference devices (SQUID's) (see Ref. [1] for a review). The model of interaction with a bath is chosen usually in such a way that the equation of motion of a system in the classical limit is of the form of that for a Brownian particle,

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = f(t), \quad \langle f(t)\rangle = 0,$$
$$\langle f(t)f(0)\rangle = 4\Gamma T\delta(t) \tag{1}$$

with the friction force $-2\Gamma\dot{q}$ proportional to velocity, and with the potential $U(q)$ that has a metastable local minimum; T is the temperature of the bath. For the system (1), the probability of escape from the metastable state via thermal activation is $W = A\exp(-\Delta U/T)$ [2], with the activation energy $\Delta U = U(q_s) - U(q_{\rm eq})$ independent of the friction coefficient Γ (q_s and $q_{\rm eq}$ are the positions of the local maximum and minimum of the potential, i.e., of the saddle point s and stable equilibrium point in the phase space). It is the prefactor A only that depends on Γ. This dependence was analyzed in Ref. [2] for strong to intermediate friction, while for weak friction, $\Gamma \lesssim |U''(q_s)|^{1/2}T/\Delta U$, it was considered in Ref. [3] (see also references therein).

The reason why the escape activation energy is independent of Γ is that dissipation and fluctuations of the system (1) are interrelated via the fluctuation-dissipation theorem and the quasistationary intrawell probability distribution $\rho(q,p)$ is close to Boltzmann one, $\rho(q,p) \approx C\exp[-E(q,p)/T]$ for $U(q_s) - E \gg T$, where $E(q,p)$ is the energy. This distribution is formed over the relaxation time t_r [$t_r = \Gamma^{-1}$ for the system (1)], and remains quasistationary in the time domain

$$t_r \ll t \ll W^{-1} \; . \tag{2}$$

The escape rate is determined by the probability to find the system near the unstable equilibrium point ($q = q_s, p \equiv \dot{q} = 0$), and therefore it is given by $\exp(-\Delta U/T)$, with an accuracy to a prefactor. In the general case of nonthermal systems, e.g., those driven by time-dependent fields, dissipation would be expected to influence not only the prefactor, but also the "activation energy" of the escape rate.

In the present paper dissipative corrections to the *arguments* of the exponentials for the escape rates of underdamped thermal-nonequilibrium classical systems are analyzed. The problem is actual for many physical systems, in particular, for weakly absorbing optically bistable systems displaying refractive bistability [4], underdamped dc-biased Josephson junctions with coexisting regimes of dc and ac current [5], and electrons trapped in a Penning trap and displaying bistability of resonantly driven cyclotron motion [6]. We note that for periodically driven systems, including the latter one, it is not energy but quasienergy [7] (see also Sec. IV below) that is conserved in the neglect of external noise and of coupling causing dissipation.

A convenient approach to the analysis of the escape rates for systems driven by weak Gaussian noise is based [8–11] on the method of optimal path. The idea goes back to Feynman [12], and a rigourous formulation was given by Wentzel' and Freidlin [13] in the problem of probability distribution of a Markov system with a positive definite matrix of diffusion coefficients (see also Refs. [14,15]). The method of optimal path makes it possible to reduce, to logarithmic accuracy, the calculation of escape probability of a noise-driven system to a solution of a boundary-value problem for a set of differential or integrodifferential equations. The explicit form of the equations depends on the form of the initial equations of motion of the system and on the shape of the power spectrum of noise, and even for white-noise-driven systems the problem is nonintegrable, generally speaking [16,17]. It is substantial, however, that if the boundary of the basin of attraction to a stable state of the initial system (in the absence of noise) is smooth the optimal path along which the system escapes (with an overwhelming probability) ends in the saddle point on this boundary [11].

The motion of an underdamped system driven by weak noise is mostly vibrations with an energy (quasienergy) E slowly varying in time. Therefore one would expect the quasistationary distribution ρ in the time range (2) to depend on coordinates and momenta of the system in terms of $E(q,p)$, primarily. The corrections proportional to $[\omega(E)t_r]^{-1}$ where $\omega(E)$ is the eigenfrequency of the vibrations with a given energy, were considered in Ref. [16(b)]. These corrections arise in the argument of the exponential, so, they can be large in absolute value even for small t_r^{-1}. We note, however, that the eigenfrequency $\omega(E)$ falls down to zero for E approaching the energy of a saddle. Therefore, the dissipative corrections should increase for such energies and one would expect them to be particulary important just for the escape rates. Moreover, since $\omega(E)$ is a nonanalytic function of the energy, a simple perturbation theory in $[\omega(E)t_r]^{-1}$ is inapplicable and the corrections to the escape rates can be nonanalytic in the dissipation parameters.

The dissipative corrections are investigated in the present paper for white-noise-driven systems of a quite general form. The zero-damping limit is considered in Sec. II. In Sec. III there is proposed a method that makes it possible to find nonanalytic corrections; it is based on the fact that the slowing down of the motion along the optimal path occurs near a saddle point, but in the vicinity of this point the equations of motion can be linearized (we note, however, that the corrections are not "local" themselves; they are determined by the motion far from the saddle point and depend crucially on the structure of the trajectories of the motion in the absence of noise). In Sec. IV the results are applied to a nonlinear oscillator bistable in a nearly resonant field, which models in particular bistability of a trapped electron investigated in Ref. [6]; to the lowest order in dissipation the escape rates for this model were found in Ref. [8]. The analytic expressions are compared with detailed numerical data for the above model. The data has been obtained by solving the boundary-value problem for the optimal paths along which the oscillator escapes from the stable states. A new generic feature of the pattern of optimal paths, the onset of caustics, is revealed.

II. ACTIVATION ENERGY OF A TRANSITION IN ZERO-DAMPING LIMIT

A. The model

We shall consider fluctuations of a dissipating noise-driven dynamical system described by the equations

$$\dot{q} = g_p - \eta\bar{q} + f_1(t),\ \ \dot{p} = -g_q - \eta\bar{p} + f_2(t)\ ,$$
$$\langle f_{1,2}\rangle = 0,\ \ \langle f_i(t)f_j(t')\rangle = \eta D B_{ij}(q,p)\delta(t-t')\ , \tag{3}$$
$$g_p \equiv \frac{\partial g}{\partial p},\ \ g_q \equiv \frac{\partial g}{\partial q}\ .$$

Here, $g \equiv g(q,p)$ is the effective dimensionless energy of the system; the dimensionless parameters η and D characterize the "strength" of dissipation and noise and are assumed small (the criterion is given below).

The dynamics of the isolated system is supposed to be time reversal, with the coordinate q and momentum p being even and odd functions of time, respectively, i.e., with $g(q,p)$ being an even function of momentum

$$g(q,p) = g(q,-p)\ . \tag{4}$$

It is dissipation that brings time irreversibility, and therefore we assume that the functions $\bar{q}(q,p)$, $\bar{p}(q,p)$ in (3) satisfy the relations

$$\bar{q}(q,p) = \bar{q}(q,-p),\ \ \bar{p}(q,p) = -\bar{p}(q,-p)\ . \tag{5}$$

Equations (4) and (5) hold for a variety of physical systems in the absence of a magnetic field, including those mentioned in the Introduction, in particular colloidal particles driven by a potential field [their dynamics is reasonably well described by Eq. (1) and is of interest for various studies in physics, cf. Ref. [18] and references therein] and a nonlinear oscillator driven by a nearly resonant force (in this case g is not energy, but quasienergy; see below). The explicit form of the functions $g, \bar{q}, \bar{p}$ can be quite complicated: unlike mechanical energy equal to the sum of potential and kinetic energies, $g(q,p)$ may not be a sum of the terms that depend on q and p separately (cf. Sec. IV).

The topology of the phase space of conservative motion of a system with two dynamical variables is comparatively simple, although even for such systems there are several patterns of evolution of the trajectories of conservative motion with the varying energy $g(q,p)$. We consider the two patterns that are most simple and most common for bistable systems. They are shown in Fig. 1 and refer to the case when both the center f and the saddle point s involved lie on the q axis. Their coordinates ($q = q_f, p = 0$ and $q = q_s, p = 0$, respectively) are given by the equation

$$g_q(q,0) = 0 \quad \text{for } q = q_f, q_s\ . \tag{6}$$

For finite damping the center is assumed to transform into an attracting focus, and the positions of the fixed points are shifted along the q and the p axis with respect to the zero-damping positions by the distances $\sim \eta^2$ and $\sim \eta$, respectively, according to (5). The generalization of the analysis given below to the case where the attractors are limit cycles instead of foci is straightforward; it is the topology of the trajectories in the vicinity of the sad-

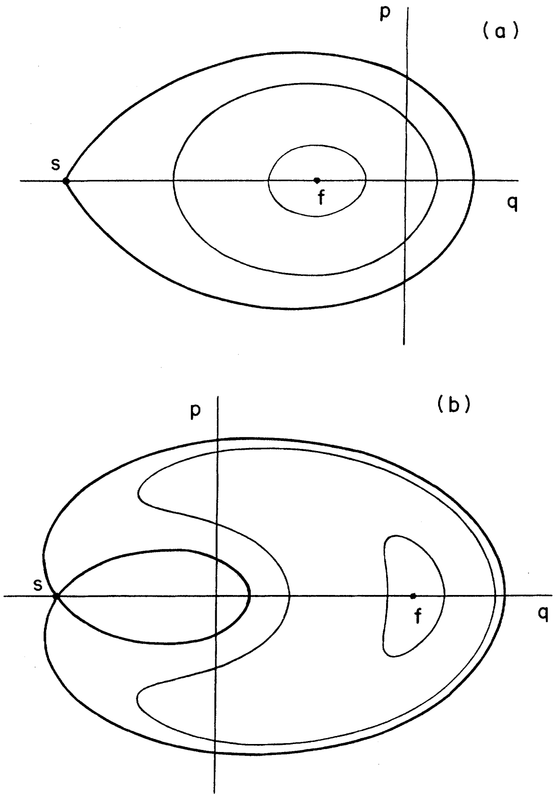

FIG. 1. Two simplest topologically different patterns of the phase trajectories of conservative motion (7); f is a center (it transforms into a stable focus for finite damping), s is a saddle point. (a) The momentum p as a function of the coordinate q has two branches for a given energy g, and the trajectories with g close to g_s pass the vicinity of the saddle point once within a cycle. (b) The momentum has four branches for g close to g_s, and the trajectories pass the vicinity of the saddle point twice within a cycle.

dle point that determines dissipative corrections to the "activation energy" of the escape from a stable state.

The pattern in Fig. 1(a) (called type-I pattern in what follows) is the most standard one; it refers, e.g., to a particle moving in a potential well, with the center and the saddle point corresponding to a minimum and maximum of the potential. The pattern in Fig. 1(b) (it is called type-II pattern below) is a bit more complicated. It refers to systems where $\partial g/\partial p$ vanishes on the trajectory not only for $p = 0$, but also for some $p \neq 0$, so that p as a function of the coordinate q as given by the equation $g(q,p) = g$ has four (or more) branches in a certain range of g. For $g = g_s$ [$g_s \equiv g(q_s, 0)$ is the value of the effective energy in the saddle point] the trajectory transforms into two closed loops, one inside the other, touching each other at the cusp point $q = q_s, p = 0$. It is quite clear that for g lying on one side of g_s the trajectories of the conservative motion are of the shape of a horseshoe as shown in Fig. 1(b), whereas for g lying on the opposite side of g_s the two types of the trajectories coexist: those inside the internal loop of the separatrix are of the shape shown in Fig. 1(a), and those outside the external loop still have a horseshoelike shape, at least for g close to g_s.

It is convenient to describe conservative motion by variables g, τ related to q, p by the equations

$$q_\tau = g_p, \quad p_\tau = -g_q, \quad \left|\frac{\partial(q,p)}{\partial(\tau,g)}\right| = 1 \; . \tag{7}$$

The variable τ is seen from (7) to be the "time" of conservative motion with a given g. It is proportional to the phase of eigenvibrations, and q, p are periodic in τ with a period $T(g) = 2\pi/\omega(g)$ [$\omega(g)$ is the frequency of eigenvibrations with a given energy g],

$$q(g,\tau+T(g)) = q(g,\tau), \quad p(g,\tau+T(g)) = p(g,\tau) \; ,$$
$$q(g,\tau) = q(g,-\tau), \quad p(g,\tau) = -p(g,-\tau) \; . \tag{8}$$

In what follows we assume that τ lies between $-\frac{1}{2}T(g)$ and $\frac{1}{2}T(g)$ (cf. Fig. 2), and we choose the origin $\tau = 0$ so that, on the trajectories in Fig. 1, $p = 0$ for $\tau = 0$ and $q(\tau = 0) - q_f$ is opposite in sign to $q_s - q_f$. Closed loops in Fig. 1 correspond to horizontal segments on the phase plane (τ, g) in Fig. 2.

When weak dissipation is allowed for in (3), the centers in Figs. 1(a) and 1(b) are transformed into stable foci and the closed loops are transformed into spirals along which the system approaches the foci. Dissipative trajectories in variables (τ, g) are of the type sketched in Fig. 2 with the arrows pointing in the opposite direction [it has been taken into account in Fig. 2 that τ varies from $-\frac{1}{2}T(g)$ to $\frac{1}{2}T(g)$ and that the period $T(g) \to \infty$ for $g \to g_s$]. Dissipation can be assumed weak provided

$$\eta \ll \Omega_s, \quad \Omega_s = |g_{qq}g_{pp}|^{1/2}_{q=q_s,p=0} \; ,$$
$$\eta \ll \omega(g) \quad \text{for } |g - g_s| \gg \eta \; , \tag{9}$$

where Ω_s is the characteristic increment or decrement of the motion near the saddle point. If (9) holds the effect of dissipation in the absence of noise comes primarily to

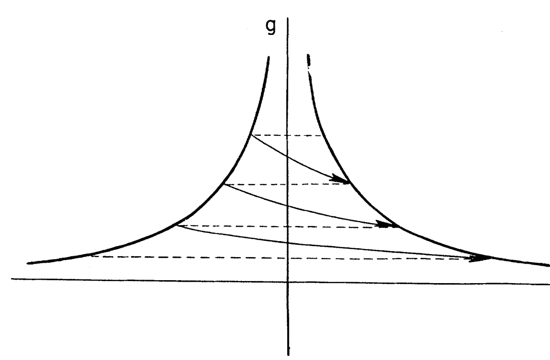

FIG. 2. The projection of an optimal path described by (27) on the (τ, g) plane. Segments of the path shown by solid lines are quasicycles. Horizontal segments (dashed lines) correspond to the conservative motion in the absence of dissipation and noise, i.e., to the closed loops in the variables (q, p) in Fig. 1.

slow variation in time of the effective energy g. The sign of $\dot{g}$ averaged over the period $T(g)$ coincides with that of $g_f - g_s$ [$g_f \equiv g(q_f, 0)$ is the value of g in a focus], and g approaches g_f over the time $\sim \eta^{-1}$.

When considering the symmetry properties of the diffusion coefficients $B_{ij}(q,p)$ in (3) we shall assume that noise itself does not break the symmetry of the system with respect to time inversion. In this case

$$B_{ij}(q,p) = B_{ij}(q,-p)(-1)^{i+j} \quad (i,j=1,2),$$
$$B_{11}B_{22} - B_{12}^2 \geq 0, \; B_{12} = B_{21} \; . \tag{10}$$

The noise-intensity parameter D is supposed to be small,

$$D \ll \eta \; , \tag{11}$$

so that D is the smallest parameter of the theory. This is because of the inequality (11) that dissipative corrections turn out exponentially large.

B. Hamilton-Jacobi equation for the optimal path

A simple and quite general way to obtain equations for the optimal path of the escape of a system driven by Gaussian noise is based [8,11] on the Feynman's idea [12] of the interrelation between the paths of the system and of the driving noise. In the particular case of a white-noise-driven system the equations can be obtained also by applying a WKB approximation to the Fokker-Planck equation for the probability density distribution $\rho(q,p)$ [14,15,19], i.e., by seeking the solution of this equation in the form

$$\rho(q,p) = \text{const} \times \exp[-R(q,p)/D] \; .$$

The distribution $\rho(q,p)$ is quasistationary in the range (2), and to the lowest order in the noise intensity D the equation for $R(q,p)$ is of the form

$$H(R_q, R_p; q, p) = 0 \; ,$$
$$H = \tfrac{1}{2}\eta\left(B_{11}R_q^2 + 2B_{12}R_qR_p + B_{22}R_p^2\right) + (g_p - \eta\bar{q})R_q - (g_q + \eta\bar{p})R_p \; . \tag{12}$$

The function $R(q,p)$ can be associated with a mechanical action of a two-dimensional auxiliary system with the coordinates q,p, and Eq. (12) is then the Hamilton-Jacobi equation. The related Hamilton equations for the "coordinates" q,p and "momenta" R_q, R_p describe the extreme paths of the system. The probability of the escape from a stable state f is given, to logarithmic accuracy, by the expression [8,9,15(b)]

$$W = \text{const}\times\exp(-R/D), \quad R \equiv R(\tilde{q}_s, \tilde{p}_s), \; R(\tilde{q}_f, \tilde{p}_f) = 0 \, , \tag{13}$$

where $\tilde{q}_{f,s}, \tilde{p}_{f,s}$ are the "true" (those allowing for dissipation) values of the coordinate and momentum of the initial system in the stable state f and in the saddle point s.

To calculate the activation energy R of the escape in the case of weak damping it is convenient to change from the variables q,p to g,τ. The Hamiltonian H (12) in these variables takes on a form

$$H = R_\tau + \eta F(R_\tau, R_g; \tau, g),$$
$$R(\tau, g) \equiv R(q(\tau,g), p(\tau,g)) \; ,$$
$$F(R_\tau, R_g; \tau, g) = JR_g + \Upsilon R_\tau + \tfrac{1}{2}\left(M_{11}R_\tau^2 + 2M_{12}R_\tau R_g + M_{22}R_g^2\right) \; , \tag{14}$$

where

$$J(\tau,g) = -\bar{q}g_q - \bar{p}g_p, \quad \Upsilon(\tau,g) = -\bar{q}\tau_q - \bar{p}\tau_p \; ,$$
$$M_{11} = \sum_{i,j} B_{ij}\frac{\partial\tau}{\partial x_i}\frac{\partial\tau}{\partial x_j}, \quad M_{12} = \sum_{i,j} B_{ij}\frac{\partial g}{\partial x_i}\frac{\partial \tau}{\partial x_j}, \tag{15}$$
$$M_{22} = \sum_{i,j} B_{ij}\frac{\partial g}{\partial x_i}\frac{\partial g}{\partial x_j} \quad (x_1 \equiv q, \; x_2 \equiv p) \; .$$

Because of the periodicity of q,p as the functions of τ the function $R(\tau,g)$ is periodic in τ as well,

$$R(\tau, g) = \sum_n R_n(g)\exp\left[in\omega(g)\tau\right] \; . \tag{16}$$

It follows from (16) in particular, that R_g is discontinuous at the boundaries $\tau = \pm T(g)/2$:

$$R_g(\tfrac{1}{2}T(g), g) = R_g(-\tfrac{1}{2}T(g), g) - \frac{dT(g)}{dg}R_\tau(\tfrac{1}{2}T(g), g) \; . \tag{17}$$

It is seen from (12), (14), and (16) that, to the lowest order in the dissipation parameter η,

$$R(\tau, g) = R_0(g) \; . \tag{18}$$

The equation for $R_0(g) \approx R_0^{(0)}(g)$ [the superscript (0) means the zeroth-order approximation in η] can be obtained by integrating the equation $H = 0$ over τ from $-\frac{1}{2}T(g)$ to $\frac{1}{2}T(g)$ (averaging over τ). It is of the form

$$\langle F\left(0, R_{0g}^{(0)}; \tau, g\right)\rangle_\tau = 0, \quad R_{0g}^{(0)} \equiv dR_0^{(0)}/dg \; ,$$
$$\langle f(\tau,g)\rangle_\tau \equiv \int_{-\frac{1}{2}T(g)}^{\frac{1}{2}T(g)} d\tau f(\tau, g) \; . \tag{19}$$

It follows from (19) with the account taken of Eqs. (14) and (15) that

$$R_0^{(0)}(g) = -2\int_{g_f}^{g} dg \frac{I(g)}{M(g)}, \quad I(g) = \langle J(\tau,g)\rangle_\tau,$$
$$M(g) = \langle M_{22}(\tau,g)\rangle_\tau \; , \tag{20}$$

and thus to zeroth order in η the activation energy of the escape

$$R \approx R^{(0)}, \quad R^{(0)} = R_0^{(0)}(g_s) \ . \tag{21}$$

The probability distribution of a nonlinear system as given by Eqs. (18) and (20) is quasi-Boltzmannian: it depends on the coordinate and momentum in terms of the effective energy g only [cf. Refs. [9,16(b),20]], but the effective "temperature"

$$\Theta(g) = -M(g)/2I(g) \tag{22}$$

is energy dependent. For a particular case of a resonantly driven nonlinear oscillator Eq. (21) goes over into the result obtained in a different way in Ref. [8]; the probability distribution was obtained in this case in yet another way in Ref. [21].

The effective temperature $\Theta(g)$ (22) is a nonanalytic function of g, in the general case. The nonanalyticity is due to the divergence of the period of eigenvibrations $T(g)$ for $g \to g_s$:

$$T(g) \approx A\Omega_s^{-1} \ln\left(|g - g_s|^{-1}\right) - 2t_s, \quad |g - g_s| \ll 1 \ , \tag{23}$$

where $A = 1$ for the trajectories of the type I [those in Fig. 1(a)] and $A = 2$ for the type-II trajectories [those in Fig. 1(b); these trajectories are obvious to pass the vicinity of the saddle point twice within a period].

For q, p lying close to the saddle point the energy $g \equiv g(q,p)$ as a function of q, p is given by the expression

$$\delta g \equiv g - g_s \approx \tfrac{1}{2} g_{qq}^{(s)} (q - q_s)^2 + \tfrac{1}{2} g_{pp}^{(s)} p^2, \ |\delta g| \ll |g_f - g_s| \tag{24}$$

[the superscript (s) means that the derivatives are calculated for $q = q_s, p = 0$]. It follows from (5), (15), (19), (20), and (24) that, if B_{ij} are regular for $q = q_s, p = 0$ (as it would be expected, generally), the functions $I(g), M(g)$ do not diverge for $g \to g_s$ although the period $T(g)$ does, and thus so do the limits of the integrals that give τ-averaged values $I(g), M(g)$ of $J(\tau, g), M_{22}(\tau, g)$ (20) (see also below). At the same time the derivatives $I_g(g), M_g(g)$ are proportional to $T(g) \propto \ln\left(|g - g_s|^{-1}\right)$ for $g \to g_s$. Therefore $d\Theta/dg$ is also proportional to $\ln\left(|g - g_s|^{-1}\right)$. It is this singularity that gives rise to nonanalytic dissipative corrections to the escape rates. We note that for thermally equilibrium systems of the type (1) the ratio $I(g)/M(g)$ is g independent, and, formally, it is because of this independence that dissipative corrections do not arise in this case.

III. METHOD OF POINCARÉ CROSS SECTION FOR THE OPTIMAL PATH OF THE ESCAPE

To the first order in η, the dissipative corrections to the function $R(\tau, g)$ far from the saddle point are seen from (12), (14), and (16) to be of the form

$$R^{(1)}(\tau, g) = R_{\rm osc}^{(1)}(\tau, g) + R_0^{(1)}(g) \ ,$$
$$R_{\rm osc}^{(1)}(\tau, g) = -\eta \sum_{n\,(\neq 0)} (2\pi i n)^{-1} \exp\left[in\omega(g)\tau\right] \langle F(0, R_{0g}^{(0)}; \tau, g) e^{-in\omega(g)\tau} \rangle_\tau \ ,$$
$$\frac{dR_0^{(1)}(g)}{dg} = -\left\langle \left(\frac{\partial F(R_\tau, R_{0g}^{(0)}; \tau, g)}{\partial R_\tau}\right)_{R_\tau = 0} \frac{\partial R_{\rm osc}^{(1)}}{\partial \tau} \right\rangle_\tau \left(\left\langle \frac{\partial F(0, R_{0g}^{(0)}; \tau, g)}{\partial R_{0g}^{(0)}} \right\rangle_\tau \right)^{-1}, \quad |g - g_s| \gg \eta \ . \tag{25}$$

The important consequence of (25) and of the properties of the kinetics of the system with respect to time inversion (4), (5), (8), (10) is that $F(0, R_{0g}; \tau, g)$ is even in τ (i.e., in the momentum p of the initial system) whereas $\partial F/\partial R_\tau$ is odd for $R_\tau = 0$, and therefore $R^{(1)}(\tau, g)$ as a whole is odd [by virtue of these arguments the term with $\partial R_{\rm osc}^{(1)}/\partial g$ dropped in the last line in (25) vanishes]. As a result,

$$R^{(1)}(0, g) = 0, \quad |g - g_s| \gg \eta \ . \tag{26}$$

The corrections to $R(0, g)$ [or, equivalently, to $R(q, p)$ for $p = 0$] for g far from g_s are of order η^2 and will be neglected in what follows.

A. General formulation

The main dissipative corrections to the activation energy of the transition R (13) are related to the slowing down of the motion near the saddle point. It is convenient to analyze them by making use of the Hamilton equations of motion that correspond to the Hamilton-Jacobi equations (12) and (14). These equations are of the form

$$\dot\tau = 1 + \eta\left(\frac{\partial F}{\partial R_\tau}\right), \quad \dot g = \eta \frac{\partial F}{\partial R_g} \ ,$$
$$\dot R_g = -\eta \frac{\partial F}{\partial g}, \quad R_\tau = -\eta F, \quad F \equiv F(R_\tau, R_g; \tau, g) \tag{27}$$

where τ, g, R_τ, R_g are the generalized time-dependent coordinates and momenta of an auxiliary system moving along the optimal path. The "coordinate" τ is fast, $\partial\tau/\partial t \simeq 1$, while g is slow. As a whole, the projection of the path described by (27) on the (τ, g) plane consists of "quasicycles" shown in Fig. 2 (we note that they are not identical in shape to the quasicycles of underdamped motion in the absence of noise, and the difference does not come just to the inversion of time, generally speaking). In the (q, p) representation the optimal path is a small-step spiral. According to (27), for

$|g - g_s| \gg \eta$ the value of τ changes by the period $T(g)$ per time $t \approx T(g) \sim \Omega_s^{-1}$. Over this time g and the "momenta" R_g, R_τ change by $\approx \eta$. As g approaches g_s the duration of motion along a quasicycle increases, and therefore the change in g, R_g, R_τ over a quasicycle increases as well, and it formally diverges for $g \to g_s$.

In the range

$$|\delta g| \sim \eta, \quad \delta g \equiv g - g_s \tag{28}$$

the perturbation theory based on averaging over a cycle is inapplicable. To analyze the motion in this range we shall make use of the fact that far from the saddle point in the phase space (q,p), i.e., for $|\tau| \lesssim \Omega_s^{-1}$, the motion is "fast" and the perturbation theory in η holds. For $|\tau| \gg \Omega_s^{-1}$ the solution of (27) can be obtained explicitly based on the fact that the system is close to the saddle point and the equations can be linearized in deviations from it.

To obtain a through solution it suffices to describe one quasicycle and then to match the neighboring quasicycles to each other. We enumerate the quasicycles and count them off from the latest one, i.e., from the semicycle along which the system arrives just to the saddle point; the numbers are therefore increasing in the direction opposite to that of time on the optimal path. For large numbers n, i.e. for g far from g_s, the solution obtained this way has to go over into that given by (18), (20), and (25). This is why it is convenient, allowing for (26), to choose the point $\tau = 0$ as a starting point of a quasicycle.

At the instant t_n when the auxiliary system passes through $\tau = 0$ in the course of the nth quasicycle, its dynamical variables (other than τ) take on some values $g(t_n), R_g(t_n)$, and

$$R_\tau(t_n) = -\eta F(R_\tau(t_n), R_g(t_n); 0, g_n) \;\; (\tau = 0) \; .$$

The problem comes now to the interrelations between $\delta g(t_n) \equiv g(t_n) - g_s, R_g(t_n)$, and t_n with different n that meet the condition $g \to g_s$ for $t \to \infty$ and the relations (18), (20), and (25) in the limit of large n. In a sense, the solution of this problem gives not only the value of the activation energy R (13), but also the logarithm of the probability distribution $-R(\tau, g)/D$ for g close to g_s. Obviously, $\delta g(t_n), R_g(t_n)$ are the values of $\delta g(t), R_g(t)$ on the optimal path in the Poincaré cross section of the four-dimensional phase space of the auxiliary system by the hypersurfaces $\tau = 0$ and $H = 0$. We note, however, that the system we are considering is *continuous*; we make use of Poincaré cross section just to match the solutions for the turns of the optimal path, and therefore the analysis differs completely from the interesting analysis of large fluctuations in systems with discrete time described by noisy maps (cf. Ref. [22]).

B. Dissipative corrections for systems with the type-I phase portrait

We start the analysis of dissipative corrections with the case where the initial dynamical system passes the vicinity of the saddle point once within a cycle of conservative motion [cf. Fig. 1(a)], so that, for $\tau = \frac{1}{2}T(g), g \approx g_s$ the momentum of the system p passes through zero for q close to its value in the saddle point q_s. It then follows from (24) that

$$g_{qq}^{(s)}(g_f - g_s) > 0 \; . \tag{29}$$

The most convenient range for matching the sections that correspond to successive quasicycles of optimal fluctuational path, and thus for obtaining equations for $\delta g(t_n), R_g(t_n), t_n$, is the vicinity of the saddle point. To be more precise, the quasicycles are matched to each other in the points where $p = 0$ near the saddle point:

$$\begin{aligned} R_{g\,n-1}^{(-)} &\approx R_{g\,n}^{(+)} - \Omega_s^{-1}\mathrm{sgn}\left(g_{qq}^{(s)}\right)|\delta g_n^{(+)}|^{-1} \\ &\qquad \times R_\tau(\tau(t_n^{(+)}), g_s + \delta g_n^{(+)}) \; , \\ \delta g_{n-1}^{(-)} &= \delta g_n^{(+)} \; , \\ \delta g_n^{(\pm)} &= \delta g_n + \eta \int_{t_n}^{t_n^{(\pm)}} dt \frac{\partial F(R_\tau, R_g; \tau, g)}{\partial R_g} \; , \\ R_{g\,n}^{(\pm)} &= R_g(t_n) - \eta \int_{t_n}^{t_n^{(\pm)}} dt \frac{\partial F(R_\tau, R_g; \tau, g)}{\partial g} \; , \end{aligned} \tag{30}$$

where $t_n^{(\pm)}$ are the instants for which τ takes on its limiting values shown in Fig. 2,

$$\tau(t_n^{(\pm)}) = \pm\tfrac{1}{2}T(\delta g_n^{(\pm)} + g_s) \tag{31}$$

(obviously, these values themselves depend on the instantaneous values of the energy g). The first equation in (30) is just the expression (17) for the discontinuity of $R_g(\tau, g)$ for $\tau \to \pm\frac{1}{2}T(g)$ with account taken of the explicit form of $T(g)$ (23) for g close to g_s. The matching (30),(31) provides the sections of neighboring quasicycles to overlap as the functions of time over finite time intervals.

The behavior of the integrands in (30) in the "dangerous" range $|t - t_n| \gg \Omega_s^{-1}$ is determined by Eqs. (14) and (15) with the account taken of the explicit form of the variables q, p as the functions of τ, g that follows from (7), (23), and (24),

$$\begin{aligned} q(\tau) - q_s &\approx |2g_{qq}^{(s)}|^{-1/2}\{\exp[-\Omega_s(|\tau| + t_s)] \\ &\qquad + \delta g \,\mathrm{sgn}(g_{qq}^{(s)}) \exp[\Omega_s(|\tau| + t_s)]\} \; , \\ p(\tau) &\approx |2g_{pp}^{(s)}|^{-1/2}\{\exp[-\Omega_s(|\tau| + t_s)] \\ &\qquad - \delta g \,\mathrm{sgn}(g_{qq}^{(s)}) \exp[\Omega_s(|\tau| + t_s)]\}\mathrm{sgn}\tau \; , \\ |\delta g| &\ll |g_f - g_s|, \qquad \exp[\Omega_s(|\tau| + t_s)] \gg 1 \end{aligned} \tag{32}$$

[we remind that $t_s + \frac{1}{2}T(g)$ is diverging logarithmically as a function of $|g - g_s|$ for g close to g_s according to (23)] and also of the explicit form of the functions $\bar{q}(q,p), \bar{p}(q,p)$:

$$\bar{q}(q,p) \approx \bar{q}_s + \bar{q}_s'(q-q_s), \quad \bar{p} \approx \bar{p}_s' p \ ,$$

$$\bar{q}_s \equiv \bar{q}(q_s,0), \ \bar{q}_s' \equiv \left(\frac{\partial \bar{q}}{\partial q}\right)_{q=q_s,p=0}, \bar{p}_s' \equiv \left(\frac{\partial \bar{p}}{\partial p}\right)_{q=q_s,p=0}, \tag{33}$$

$$|q-q_s| \ll |q_f - q_s|, \quad |p| \ll \max_\tau |p(\tau, g_s)| \ .$$

According to (32) and (33), the functions $\partial F/\partial g$, $\partial F/\partial R_\tau$ near the saddle point contain "fast" terms proportional to $\exp[\pm\Omega_s(|\tau|+t_s)]$; these terms are $\sim \eta^{1/2}$ for $|\delta g| \sim \eta$, so, this is $\eta^{1/2}$ (not η) that is the parameter of the perturbation theory. However, remarkably, because of the symmetry properties and of the discontinuity of R_g at the boundary $\tau = \pm\frac{1}{2}T(g)$ (17) these terms drop out from the set of matching equations for $\delta g(t_n), R_g(t_n)$. These are the slowly varying terms in $\partial F/\partial g, \partial F/\partial R_g$ in the vicinity of the saddle point that give the main dissipative corrections to $\delta g(t_n), R_g(t_n)$. It follows from (30) that these corrections are $\sim \eta T(g) \sim \eta \ln \eta$.

The details of the perturbation theory are given in the Appendix. The resulting expression for the activation energy R that allows for the dissipative correction ΔR follows from (A1), (A10), and (A13) to be of the form

$$R \approx R^{(0)} + \Delta R, \quad \Delta R = K\eta^2 |\ln \eta| \ ,$$

$$K = \tfrac{1}{4} I^2(g_s) M^{-1}(g_s) \Omega_s^{-1} \times [(\bar{q}_s' + \bar{p}_s')\mathrm{sgn} g_{qq}^{(s)} + I(g_s) M^{-1}(g_s)(B_{11}^{(s)} g_{qq}^{(s)} + B_{22}^{(s)} g_{pp}^{(s)})] \ , \tag{34}$$

$$B_{ij}^{(s)} \equiv B_{ij}(q_s,0) \ .$$

The dissipative correction ΔR (34) for a system with the type-I phase portrait is proportional to $\eta^2|\ln\eta|$, i.e., (i) there is no correction of the first order in the damping parameter η, and (ii) the correction is nonanalytic in η. To gain some insight into the origin of the structure of the expression (34) we note first that the variation of g is quadratic in the velocity of the variables of the initial system on the optimal path: $\dot{g}$ is proportional to the terms $g_p = \partial q/\partial \tau$, $g_p = -\partial p/\partial \tau$. Therefore the main contribution to the change of δg over a quasicycle, $\delta g(t_{n-1}) - \delta g(t_n)$, comes from the range of the fast motion far from the saddle point; this change is of order η. The contribution from the slow motion in the vicinity of the saddle point is of order $\eta\Omega_s^{-1}|\delta g_n^{(\pm)} \ln|\delta g_n^{(\pm)}||$; this is because the duration of the slow motion is $\sim \Omega_s^{-1}|\ln|\delta g_n^{(\pm)}||$, and the smooth component of the $\dot{g}$ is $\sim \eta\delta g_n^{(\pm)}$ according to (15), (24), (27), and (32) [near the saddle point $p^2 \sim |\delta g_n^{(\pm)}|, (q-q_s)^2 \sim |\delta g_n^{(\pm)}|$].

The variation of R_g, on the other hand, is not related directly to the initial system being in motion, and therefore R_g is changed most strongly just in the vicinity of the saddle point, and the total change over a quasicycle makes $\sim \eta|\ln|\delta g_n^{(\pm)}||$. In the range $|\delta g| \sim \eta$ we can replace $\ln|\delta g_n^{(\pm)}|$ by $\ln \eta$, i.e., to the lowest order of the perturbation theory, the logarithm can be assumed independent of δg. At the same time, for $|\delta g| \gg \eta$ the expression for $R_g^{(0)}$ that follows from (15), (20), and (26) gives $R_g(0,g)$ to the accuracy η^2, and in particular the term $\delta g \ln|\delta g|$ in this expression is accurate to η^2. It would be expected from scaling arguments that Eqs.(20) and (26) for $R_g(0,g)$ hold for all δg up to the last crossing of the axis $\tau = 0$ by the optimal path. These arguments are confirmed by the detailed calculations given in the Appendix.

The above arguments are not applicable to the contribution to the activation energy R from the last (zeroth) semicycle that arrives right to the saddle point. It is of the form

$$\delta R = \int_{t_0}^{\infty} (R_\tau \dot{\tau} + R_g \dot{g})\, dt \tag{35}$$

(we have allowed for the fact that it takes an infinitely long time to reach a saddle point moving along the optimal path). Here, the main term in R_τ is seen from (14) and (27) to be $-\eta R_g(J + \frac{1}{2}M_{22}R_g)$. The integral of this expression over τ (35) between $\tau = 0$ and $\tau = T/2$ ($T \to \infty$ for $g \to g_s$) vanishes, according to (19)and (20) if $R_g = R_{0\,g}^{(0)}(g_s)$ and $g = g_s$ in $J(\tau,g), M(\tau,g)$, i.e., the integral is equal to zero to the first order in η. The value of R_g in the second term remains nearly constant for the fast-motion range where g becomes equal to g_s to the first order in η. Since $R_g(t_0) = R_{0\,g}^{(0)}$ as given by (20), the second integral in (35) over the fast-motion region is equal to

$$-2\int_{g(t_0)}^{g_s} dg I(g_s)/M(g_s)$$

to accuracy $\sim \eta^2 \ln \eta$, and as a whole, there are no dissipative corrections of order η to the activation energy R (21). The corrections are due to the slow motion in the vicinity of the saddle point in (35) and are therefore of order $\eta^2|\ln\eta|$, just as has been written in (34).

The coefficient K in (34) is determined by the values of both the diffusion ($B_{11}^{(s)}, B_{22}^{(s)}$) and drift ($\bar{q}', \bar{p}'$) coefficients in the saddle point. The sign of K depends on a particular form of the parameters, and thus dissipation can both promote (for $K < 0$) and hinder (for $K > 0$) escape from a metastable state in the case of the type-I phase portrait of the conservative motion.

C. Dissipative corrections for systems with the type-II phase portrait

If the phase trajectories surrounding a metastable state of a system look in the zero-damping limit as the horseshoelike trajectories in Fig. 1(b), i.e., if

$$g_{qq}^{(s)}(g_f - g_s) < 0$$

the dissipative corrections to the activation energy of escape turn out to be parametrically larger than those for the case of the type-I phase portrait. The detailed analysis of the range $|g - g_s| \sim \eta$ can be fulfilled along the lines described above and in the Appendix. The result can be understood, however, from the qualitative arguments presented below.

Just as in the case considered above, in the present case the optimal path of fluctuational escape from a metastable state is a small-step spiral determined by Eqs.(12) and (14) or Eqs. (27). To the lowest order in η the approximation (18) holds for this spiral, and the action $R(\tau, g)$ is described by Eqs. (19) and (20). However, the averaging over a period (19) can be viewed now as that over the external and internal parts of a horseshoelike trajectory [cf. Fig. 1(b)], and in contrast to the former case we have an option of setting τ equal to zero for $p = 0$ on either the external or internal part of the trajectories. If we set $\tau = 0$ on the external part then

$$\langle f(\tau, g)\rangle_\tau = \langle f(\tau, g)\rangle_e + \langle f(\tau, g)\rangle_i \ , \tag{36}$$

$$\langle f(\tau, g)\rangle_e = \int_{-\tau_b}^{\tau_b} d\tau f(\tau, g),$$
$$\langle f(\tau, g)\rangle_i = \int_{\tau_b}^{T(g)-\tau_b} d\tau f(\tau, g) \ , \tag{37}$$
$$\left(\frac{\partial p(\tau, g)}{\partial \tau}\right)_{\tau_b} = 0 \ .$$

To consider the equally plausible choice, $\tau = 0$ on the internal part of a quasicycle, one should replace τ_b by $\frac{1}{2}T - \tau_b$ in (37). The boundary τ_b between the parts is chosen conditionally to lie at the point $\partial p/\partial \tau = 0$ with minimal $|p|$, where the two "teeth" of the horse-shoe-like trajectory come most closely to each other.

According to (36) the functions $I(g), M(g)$ in (20) are given by the expressions

$$I(g) = I_e(g) + I_i(g), \quad M(g) = M_e(g) + M_i(g) \ , \tag{38}$$

where I_e and I_i, M_e and M_i are the values of $J(\tau, g)$ and $M_{22}(\tau, g)$ (15) averaged over the external and internal parts of a cycle, respectively. Because of the relation

$$\frac{\partial F(R_\tau, R_{0g}; \tau, g)}{\partial R_\tau} = -\frac{\partial F(R_\tau, R_{0g}; -\tau, g)}{\partial R_\tau} \quad \text{for } R_\tau = 0$$

that follows from the symmetry properties of the system with respect to time inversion there are no corrections $\sim \eta$ to $R(0, g)$ far from the saddle point [cf. (25) and 26)] irrespective of a chosen option for the point from which τ is counted off. The analysis similar to that given in the Appendix proves that there are no corrections $\sim \eta$ to $R(0, g)$ up to the last crossing of the axis $p = 0$ by the optimal path. Therefore, just as in the former case of the type-I phase portrait, the main dissipative corrections come from the last section of the optimal path that lies between the last crossing of the axis $p = 0$ where $\tau = 0$ and the saddle point.

The contribution of the last section to the escape activation energy R is given by Eq. (35). The second term in (35), when added to the value $R(0, g(t_0))$ of $R(\tau, g)$ in the point t_0 on the optimal path gives $R^{(0)}$ (21) to the first order in η as in the case considered in Sec. III B. However, in contrast to the latter case, the first integral does not vanish to the first order in η

$$\int_{t_0}^{\infty} dt R_\tau \dot{\tau} \approx -\left[\eta \int_0^{\frac{1}{4}T(g)} d\tau R_g \left(J + \tfrac{1}{2}M_{22}R_g\right)\right]_{g\to g_s} ,$$

$$R_g(g_s) \approx -2\left[I_e(g_s) + I_i(g_s)\right] / \left[M_e(g_s) + M_i(g_s)\right] \tag{39}$$

[we have taken into account that for $g \to g_s$ the duration of motion along the both external and internal parts of the path is $\approx \frac{1}{2}T(g)$]. It follows from (36)–(39) that to the first order in η

$$R \approx R^{(0)} + \eta R^{(1)},$$

$$R^{(1)} = -\left| \frac{I_e(g_s) + I_i(g_s)}{[M_e(g_s) + M_i(g_s)]^2} \times [M_i(g_s)I_e(g_s) - M_e(g_s)I_i(g_s)] \right| . \tag{40}$$

The reason for the correction $R^{(1)}$ to be negative is quite obvious: the integral (39) taken along the external part of the last semicycle is opposite in sign to that taken along the internal part. The true optimal path corresponds to minimal escape activation energy; therefore, the last crossing of the axis $p = 0$ ($\tau = 0$) happens just on that part of the quasicycle for which the dissipative correction is negative. Thus, in contrast to the case of the type-I phase portrait shown in Fig. 1(a), in the case of horseshoelike trajectories of conservative motion there arises a first-order dissipative correction to the activation energy of escape from a metastable state, and this correction is negative. Its magnitude depends on the relationship between the coefficients of drift and diffusion over energy averaged over external and internal parts of the path for $g \to g_s$.

IV. ACTIVATION ENERGIES OF FLUCTUATIONAL TRANSITIONS FOR AN UNDERDAMPED PERIODICALLY DRIVEN NONLINEAR OSCILLATOR

A system where both types of dissipative corrections to the probability of escape from a metastable state can be immediately investigated experimentally is nonlinear oscillator bistable in a nearly resonant driving field. The system has been mentioned above to model bistable forced cyclotron vibrations of an electron in a Penning trap [6]. Fluctuational transitions between attractors of a periodically driven Duffing oscillator were considered and the activation energies were obtained to zeroth order in dissipation in Ref. [8]. We note that the bistability we discuss *emerges* because of the field, in the absence of the field the oscillator is assumed to have one stable state; the effect of periodic field on the probability of thermally activated escape from a metastable potential well was considered in Ref. [23].

The motion of the oscillator in the absence of dissipation and fluctuations is described by the equation

$$\frac{d^2x}{dt'^2} + \omega_0^2 x + \gamma x^3 = h\cos\omega_h t', \quad |\omega_0 - \omega_h| \ll \omega_h \ . \tag{41}$$

Here, x is the oscillator coordinate, ω_0 is the eigenfrequency of small-amplitude vibrations, h is the amplitude of a nearly resonant field.

If, on one hand, the field is not too large so that the amplitude of the vibrations is very much smaller than the characteristic length $|\omega_0^2/\gamma|^{1/2}$, and, on the other hand, coupling to a bath which gives rise to dissipation and fluctuations of the oscillator is linear in x and weak so that the oscillator is underdamped, then, for a quite general shape of the density of states of the bath weighted with the coupling parameters, the motion of the oscillator is mostly fast vibrations at frequency ω_h with slowly varying amplitude and phase. It is convenient to describe this motion in terms of smooth variables q, p and "slow" time t,

$$\begin{aligned} q = &(3|\gamma|/8\omega_h|\omega_h - \omega_0|)^{1/2} \\ &\times \left(x\cos\omega_h t' - \omega_h^{-1}\frac{dx}{dt'}\sin\omega_h t' \right)\operatorname{sgn}(\omega_h - \omega_0) \;, \\ p = &-(3|\gamma|\omega_h/8|\omega_h - \omega_0|)^{1/2} \\ &\times \left(x\sin\omega_h t' + \omega_h^{-1}\frac{dx}{dt'}\cos\omega_h t' \right)\operatorname{sgn}(\omega_h - \omega_0) \\ t = &|\omega_h - \omega_0| t' \;. \end{aligned} \tag{42}$$

The equations for q, p that follow from (41) and (42) with account taken of dissipation and fluctuations are of the form of Eq. (3) [8], with

$$\begin{aligned} &g(q,p) = \tfrac{1}{4}(q^2 + p^2 - 1)^2 - q\beta^{1/2}, \\ &\beta = 3|\gamma|h^2/32\omega_h^3|\omega_h - \omega_0|^3 \;, \\ &\bar{q}(q,p) = q, \;\; \bar{p}(q,p) = p, \\ &B_{11} = B_{22} = 1, \;\; B_{12} = B_{21} = 0 \;. \end{aligned} \tag{43}$$

The effective energy g (43) is just the dimensionless quasienergy of the periodically driven oscillator [21]. It contains one parameter, β, which is obvious to be a dimensionless intensity of the driving field. If dissipation of the oscillator is described by a linear friction force $-2\Gamma dx/dt'$ as in Eq. (1) then the dissipation parameter η in Eqs. (3) and (43) is equal to $\Gamma/|\omega_h - \omega_0|$. However, Eqs. (3) and (43) hold for a much more broad class of coupling mechanisms than those assumed in (1): the delay of relaxation has been neglected on the *slow-time* scale $|\omega_h - \omega_0|^{-1}, \Gamma^{-1}$ only (see Ref. [8] for details). Also, random force driving the oscillator is assumed δ correlated on the same time scale; if noise is due to thermal fluctuations in the bath, the characteristic noise intensity D in (3) is equal to $3|\gamma|T/4\omega_h^3|\omega_h - \omega_0|$.

In the weak-damping limit the oscillator is bistable in the range of the amplitude of the force h and the frequency detuning $\omega_h - \omega_0$ encompassed by the conditions

$$\eta^2 < \beta < \tfrac{4}{27} + \tfrac{1}{3}\eta^2 \quad (\eta \ll 1) \;.$$

The coexisting attractors correspond to forced vibrations with different amplitudes and phases. Phase portraits in the slow variables q, p for the ranges of attraction to the small- and large-amplitude states (the states 1 and 2) in the limit $\eta \to 0$ look like those in Fig. 1(a) and Fig. 1(b), respectively. The quasienergy g takes on its minimal value in the large-amplitude state, while in the small-amplitude one $g(q,p)$ has a local maximum. The value $g = g_s$ of the quasienergy in the unstable steady state lies in between the values of g in the attractors.

To zeroth order in dissipation the escape activation energies $R \approx R^{(0)}$ for the oscillator are the functions of one dimensionless parameter β (the dimensionless field intensity) only, and, for the small-amplitude state, $R^{(0)}$ decreases with increasing β and becomes zero for $\beta = 4/27$ where this state merges with the saddle state, whereas for the large-amplitude state the dependence of $R^{(0)}$ on β is opposite [8].

Dissipative corrections to the activation energies for the two attractors are qualitatively different as it follows from the results of Secs. III B and III C. In the case of the large-amplitude attractor the correction is described by Eq. (40) and is of the order of η whereas in the case of the small-amplitude attractor the main correction is of the order of $\eta^2|\ln\eta|$. The corresponding coefficients $R^{(1)}$ (40) and K (34) can be calculated numerically as the functions of β (which is the only parameter of the problem). The results are shown in Fig. 3. It is seen from Fig. 3 that the dissipative corrections decrease in absolute values with increasing β for the both attractors (the zero-dissipation term $R^{(0)}$ increases with β for the large-amplitude attractor).

Analytic expressions for $K, R^{(1)}$ can be obtained near bifurcation points $\beta = \eta^2$ and $\beta = \frac{4}{27} + \frac{1}{3}\eta^2$. For small fields the small-amplitude (SA) attractor is much more stable with respect to fluctuations than the large-amplitude (LA) one that merges with the saddle point for $\beta = \eta^2$. The values of $R^{(0)}$ for these attractors, $R^{(0)}_{\mathrm{SA}}$ and $R^{(0)}_{\mathrm{LA}}$, respectively, are given by the expressions

$$R^{(0)}_{\mathrm{SA}} \approx 1, \quad R^{(0)}_{\mathrm{LA}} \approx 4\beta^{1/2}, \quad \eta^2 \ll \beta \ll 1 \;, \tag{44}$$

while, according to Eqs. (34), (40) and (43)

$$\begin{aligned} &K \equiv K_{\mathrm{SA}} \approx \frac{2^{1/2}\pi^3}{64}\beta^{-3/4}, \\ &R^{(1)} \equiv R^{(1)}_{\mathrm{LA}} \approx -2\pi, \;\; \beta \ll 1 \;. \end{aligned} \tag{45}$$

The value of K increases sharply with decreasing β. However, the dissipative correction $K\eta^2|\ln\eta|$ remains small compared to R_{SA} up to the bifurcation point $\beta = \eta^2$ where the small-amplitude state becomes the only stable state of the system. At the same time, $\eta R^{(1)}_{\mathrm{LA}} \sim R^{(0)}_{\mathrm{LA}}$ for $\beta \sim \eta^2$, and the perturbation theory is thus inapplicable. This would be expected to happen, since the motion in the vicinity of the large-amplitude attractor cannot be regarded underdamped for $\beta \sim \eta^2$ where this attractor is close to the saddle point in the phase space; a theory of the escape from a metastable state close to an unstable one was considered earlier [24].

Near the bifurcation point $\beta \approx \frac{4}{27}$ where the small-amplitude state disappears one has

$$R_{\mathrm{SA}}^{(0)} \approx 9\left(\tfrac{4}{27}-\beta\right)^{3/2}\ ,\quad K \equiv K_{\mathrm{SA}} \approx \xi_0\left(\tfrac{4}{27}-\beta\right)^{3/2}\ , \qquad (46)$$

$$\xi_0 \approx 9.5 \quad (\eta^2 \ll \tfrac{4}{27}-\beta \ll 1)\ ,$$

and, remarkably, the nonanalytic correction turns out to vanish as rapidly as the main term for β approaching the bifurcation value. This is a feature of a particular form of the coefficients in K as given by (34) for the model (43); in the general case K is simply proportional to the distance to a bifurcation point in the parameter space. In the bifurcation range $\frac{4}{27}-\beta \sim \eta^2$ Eqs. (21) and (46) are inapplicable, the term $R_{\mathrm{SA}}^{(0)}$ (46) is of order η^3, i.e., it is smaller than the dissipative corrections $\sim \eta^2$ omitted in (34). We note that the expression for the activation energy of escape in the vicinity of a bifurcation point [24] is given formally by the expression for $R_{\mathrm{SA}}^{(0)}$ (46) with $\frac{4}{27}-\beta$ replaced by $\frac{4}{27}-\beta+\frac{1}{3}\eta^2$.

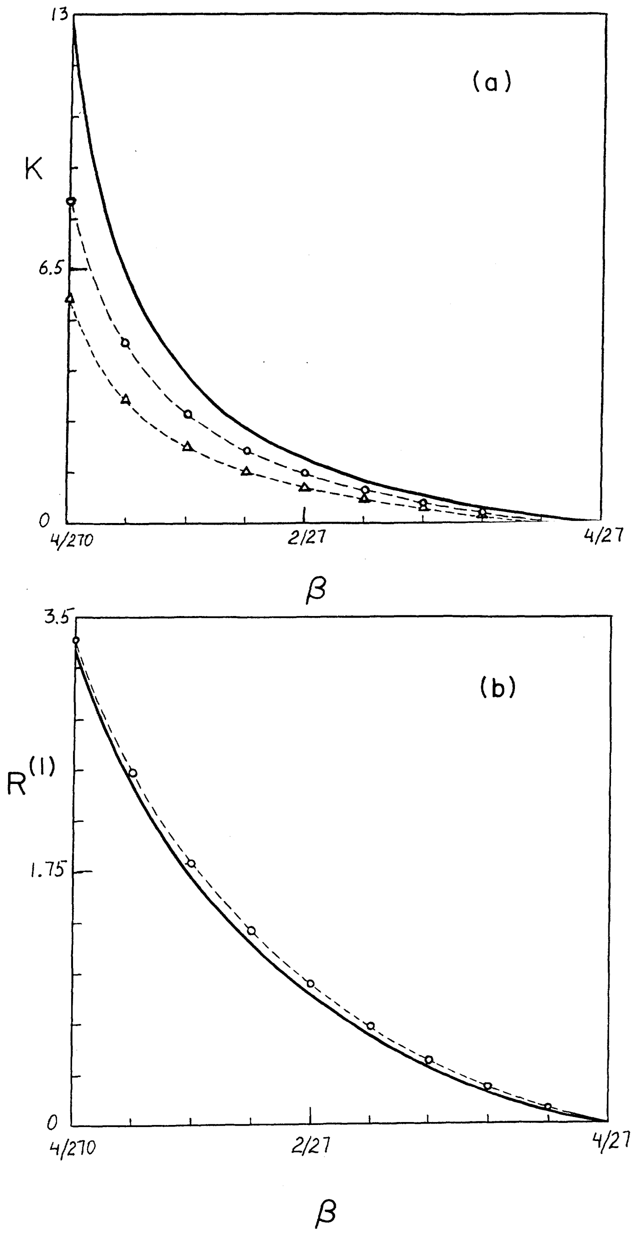

FIG. 3. Reduced dissipative corrections to the activation energies of escape from the stable states of (a) small-amplitude and (b) large-amplitude forced vibrations of a nonlinear oscillator. Full curves: the coefficients K (34) and $R^{(1)}$ (40) as the functions of the reduced intensity of the driving field β. The circles ($\eta = 0.001$) and triangles ($\eta = 0.01$) show the results for $K, R^{(1)}$ obtained by solving the boundary-value problem (27′) numerically, subtracting from the data the values $R^{(0)}$ (21) of the activation energies in the neglect of dissipation and then dividing by (a) $\eta^2|\ln\eta|$ and (b) η.

A general approach to the calculation of the activation energies of fluctuational transitions is based on numerical analysis of the Hamilton equations (27) in the variables τ, g, R_τ, R_g or (what is often more convenient) in the variables q, p, R_q, R_p:

$$\dot q = \frac{\partial H}{\partial R_q},\quad \dot p = \frac{\partial H}{\partial R_p},\quad \dot R_q = -\frac{\partial H}{\partial q},\quad \dot R_p = -\frac{\partial H}{\partial p}\ . \qquad (27')$$

Numerical analysis of the equations of this sort in the context of the probability density distribution of a system without detailed balance was first done by Ludwig [14]. In the context of the escape, numerical analysis was performed by Kautz [20] for a white-noise-driven dc-biased Josephson junction, with the coexisting stable states corresponding to the stable dc and ac regimes.

To find activation energy of the escape it is necessary to solve the boundary-value problem for Eqs. (27′) with $q \to q_f, p \to p_f$ for $t \to -\infty$, and $q \to q_s, p \to p_s$ for $t \to \infty$ (q_f, p_f and q_s, p_s are the values of q, p in a metastable focus and in a saddle point). In what follows we present the results of the numerical analysis of the activation energies and extreme paths for a resonantly driven nonlinear oscillator. The analysis has made it possible to test the above analytical expressions for dissipative corrections and also to find a new type of singularities inherent to the pattern of the extreme paths in systems without detailed balance.

The extreme paths emanating from a focus f make up a single-parameter set in the case of a white-noise-driven system with two dynamical variables (the more general case was considered in Ref. [11]; see also Ref. [25] for a review). This can be easily understood by noting that in the immediate vicinity of a focus Eqs. (27′) can be linearized in

$$x_1 = q - q_f,\quad x_2 = p - p_f\ . \qquad (47)$$

The solution of (12), (27′) for small $|x_{1,2}|$ is of the form

$$R(q,p) = \tfrac{1}{2}\sum_{i,j=1,2} A_{ij}x_i x_j,\quad |x_{1,2}| \to 0 \qquad (48)$$

where the coefficients of the matrix $\|\ A\ \|$ are given by the equations

$$\sum_{m,n}\left[\frac{\partial^2 H}{\partial R_{x_m}\partial R_{x_n}}\right]_f A_{mi}A_{nj} + \sum_m \left[\frac{\partial^2 H}{\partial x_i \partial R_{x_m}}\right]_f A_{mj} + \sum_m \left[\frac{\partial^2 H}{\partial x_j \partial R_{x_m}}\right]_f A_{mi} = -\left[\frac{\partial^2 H}{\partial x_i \partial x_j}\right]_f$$

$$(i,j = 1,2) \qquad (49)$$

(the subscript f means that the derivatives are calculated for $x_1 = x_2 = R_{x_1} = R_{x_2} = 0$). By implying (47)–(49) the initial conditions for the extreme paths (27′) can be chosen at finite small x_1, x_2 and finite t with $R_q(t), R_p(t)$ determined from (48) and (49) (cf. Ref. [14]). For $t \to -\infty$ a path parametrized in this way approaches $x_1 = x_2 = 0$ (i.e., $q \to q_f, p \to p_f$). All the points q, p, R_q, R_p a path has passed through in this way can be equally chosen as the initial ones for the same path (this provides a basis for the numerical test of the initially chosen x_1, x_2 being sufficiently small). Different paths can be parametrized by the distance along a transversal to the paths. The problem of the calculation of the activation energy gets thus reduced to that of the determination of the value of the parameter for which an extreme path arrives to a saddle point, to a given accuracy.

Numerical results for the activation energies $R_{\rm SA}$ and $R_{\rm LA}$ of the transitions from the coexisting small-amplitude and large-amplitude states of forced vibrations of the oscillator (41) are shown in Fig. 3. The quantities plotted immediately are the differences between the values of $R_{\rm SA}, R_{\rm LA}$ obtained numerically and the quantities $R^{(0)}_{\rm SA}, R^{(0)}_{\rm LA}$ (21) that give the activation energies to the zeroth order in dissipation. It is obvious from Fig. 3 that the numerical data are in very good qualitative and quantitative agreement with the analytical results for dissipative corrections. These are the *coefficients at small quantities* that are compared: e.g., for $\eta = 10^{-3}$ the dissipative correction $K\eta^2|\ln\eta|$ makes $\lesssim 10^{-5}R^{(0)}$. The data confirm that for the pattern of conservative trajectories of the type shown in Fig. 1(a) there are no first-order dissipative corrections, and the corrections are nonanalytic in the dissipation parameter η: the value of $|\ln\eta|$ is of the order of 7 for $\eta = 10^{-3}$, whereas the asymptotic expression (34) for the dissipative correction *divided* by $\eta^2|\ln\eta|$ differs from the numerical data by less than 40%. For the pattern in Fig. 1(b) the dissipative correction is linear in η indeed and negative.

A sample of the paths emanating from the small-amplitude stable state of an underdamped oscillator is shown in Fig. 4. The paths are of the spiral shape close to the focus, but far away from the focus the pattern of the paths is much more sophisticated. The features obvious immediately on inspection of Fig. 4 are self-crossing of the paths and the onset of caustics. These are not of course the "true" paths of the auxiliary system $q(t), p(t), R_q(t), R_p(t)$ but their projections on the (q, p) plane that are self-crossing, and caustics are the projections of the folds of the integration surface. The self-crossing is related to the lack of detailed balance in a system [16].

Onset of caustics is a *generic* feature of the sets of optimal paths $q(t), p(t)$ that describe large fluctuations forming the tails of the stationary probability distribution of systems without detailed balance (the pattern of optimal paths in the problem of time-dependent probability density of a transition from one point of the phase space into another displays caustics even in systems with detailed balance [26]). Of course, caustics are the features of the *sets*, not of individual paths. There are several problems posed by the onset of caustics, in particular, the position of caustics and of their end points. The most interesting ones are certainly related to physical interpretation of caustics for mono- and multistable systems: a caustic is a "mirror" that reflects optimal paths; therefore, in the simple-minded picture discussed above, a system arrives to neighboring points on the opposite sides of a caustic along completely different paths, which has to be compatible with the continuity of $R(q, p)$. This and other problems will be addressed in a forthcoming paper.

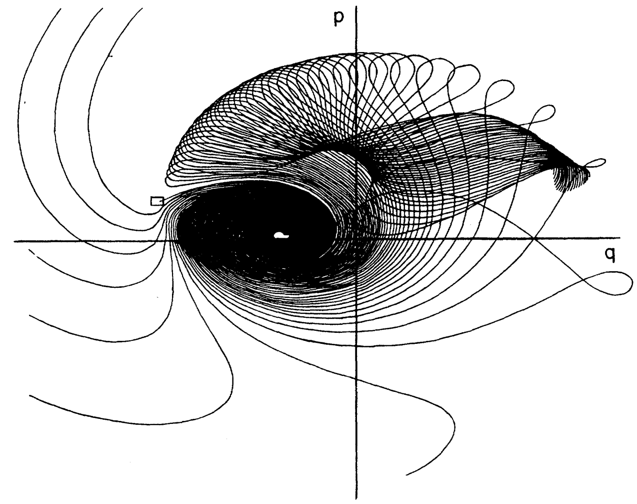

FIG. 4. A sample of the extreme paths of a nearly resonantly driven nonlinear oscillator, with clearly seen caustics. The saddle point is singled out with a rectangle.

V. CONCLUSIONS

It follows from the results of the present paper that, in underdamped thermally nonequilibrium systems, there are exponentially large dissipative corrections to the escape probabilities. The structure of the expression for the correction to the activation energy of escape depends crucially on the pattern of trajectories of conservative motion close to the saddle point. For the pattern shown in Fig. 1(a) (the trajectories pass the vicinity of the saddle point once within a cycle) the correction is nonanalytic in the dissipation parameter η ($\sim \eta^2\ln\eta$), whereas for the pattern of the type in Fig. 1(b) (the trajectories pass the vicinity of the saddle point twice within a cycle and therefore the momentum of the system as a function of the coordinate has four or more branching points) the correction is of order η and always negative, i.e., dissipation "promotes" escape.

The onset of relatively large dissipative corrections is related to the slowing down of the motion of a system in the vicinity of a saddle point. However, the corrections are not determined by local characteristics of the motion near the saddle point; it is the motion with the *energies* (quasienergies) close to the energy of the saddle point that contributes to the corrections. At the same time, the evaluation of the corrections is based substantially on the fact that, in the vicinity of a saddle point, the equations of motion can be linearized and successive sections of the optimal path can be matched to each other. The method is not limited to white-noise-driven systems. It can be generalized to some other classical fluctuating systems, and also to some quantum systems.

We note in conclusion that, since dissipative corrections to the escape probabilities are comparatively large, they can be investigated experimentally for various thermally nonequilibrium systems and for electrons in Penning traps [6] in particular, and thus the qualitative difference between the corrections for the different types of the pattern of the trajectories of conservative motion can be observed.

APPENDIX

To find $\tau(t), g(t)$, and $R_g(t)$ on the optimal path within one quasicycle for the effective energy g close to its saddle-point value g_s one can expand the derivatives $\partial F/\partial R_\tau, \partial F/\partial R_g, \partial F/\partial g$ in the equations of motion (27) in $R_\tau(t)$ and in the deviations $\tau(t) - (t - t_n), g(t) - g(t_n), R_g(t) - R_g(t_n)$ (at $t = t_n$ the trajectory passes through the point $\tau = 0$ in the nth quasicycle). It is the range $|t - t_n| \gg \Omega_s^{-1}$ where relatively large (nonanalytic in η) corrections to $\tau(t), g(t), R_g(t)$ arise. The coefficients of the function F in this range are determined by Eqs. (10), (15), (32), and (33) in the explicit form. In particular,

$$
\begin{aligned}
J \approx & -\bar{q}_s|g_{qq}^{(s)}/2|^{1/2}[e^{-\lambda} + \delta g \,\mathrm{sgn}(g_{qq}^{(s)})e^{\lambda}] \\
& -\tfrac{1}{2}(\bar{q}_s' - \bar{p}_s')\,[e^{-2\lambda} + (\delta g)^2 e^{2\lambda}] \\
& -(\bar{q}_s' + \bar{p}_s')\,\delta g \,\mathrm{sgn}(g_{qq}^{(s)}) ,
\end{aligned}
$$

$$
\lambda \equiv \Omega_s(|\tau| + t_s), \quad \delta g \equiv g - g_s , \tag{A1}
$$

$$
\begin{aligned}
M_{22} \approx & \tfrac{1}{2}\mathrm{sgn}(g_{qq}^{(s)})(B_{11}^{(s)} g_{qq}^{(s)} - B_{22}^{(s)} g_{pp}^{(s)}) \\
& \times [e^{-2\lambda} + (\delta g)^2 e^{2\lambda}] \\
& + \delta g (B_{11}^{(s)} g_{qq}^{(s)} + B_{22}^{(s)} g_{pp}^{(s)}), \quad B_{ij}^{(s)} \equiv B_{ij}^{(s)}(q_s, 0) .
\end{aligned}
$$

Near the saddle point $e^{-\lambda} \sim |\delta g|^{1/2} \sim \eta^{1/2}$ [cf. (23)] and therefore $J \sim |\delta g|^{1/2} \sim \eta^{1/2}, M_{22} \sim |\delta g| \sim \eta$. It follows then from (27) that $R_\tau \sim \eta^{3/2}$, and the terms $\sim R_\tau$ can be neglected when one seeks the lowest-order dissipative corrections to g, R.

To analyze the corrections for large $|t - t_n|$ it is convenient to change from the differential equations (27) to the integral ones:

$$
\begin{aligned}
\Delta g_n(t) &\equiv g(t) - g(t_n) = \eta \int_{t_n}^{t} dt \frac{\partial F}{\partial R_g} , \\
\Delta R_{gn}(t) &\equiv R_g(t) - R_g(t_n) = -\eta \int_{t_n}^{t} dt \frac{\partial F}{\partial g}
\end{aligned}
\tag{A2}
$$

[Eq. (27) for $\tau(t)$ can be transformed in a similar way]. The coefficients at R_g, R_τ in F as given by (14) and (15) are analytic in g [cf. (A1)]. At the same time, they contain the terms that diverge formally for $|\tau| \to \infty$. Just these terms are of primary importance close to the saddle point [the period $T(g) \to \infty$ for $g \to g_s$]; they give rise to nonanalytic in η corrections to R.

To the lowest order in η the values of g, R_g, τ in $\partial F/\partial R_g$ in (A2) can be replaced by $g_s, R_g(t_n), t - t_n$, respectively. It follows from (A1) and from the explicit form of the function F (14) that $\partial F/\partial R_g \sim \eta^{1/2}$ for $|t - t_n| \gg \Omega_s^{-1}$, and therefore the main contribution to $\Delta g_n(t)$ for large $|t - t_n|$ in (A2) comes from the integration over a range $\sim \Omega_s^{-1}$ about t_n . This contribution is given by the expression

$$
\begin{aligned}
&\Delta g_n^{(1)} = \tfrac{1}{2}\eta\,[I(g_s) + R_g(t_n)M(g_s)]\,\mathrm{sgn}(t - t_n), \\
&|t - t_n| \gg \Omega_s^{-1} .
\end{aligned}
\tag{A3}
$$

On the other hand, the main contribution to $\Delta R_{gn}(t)$ comes from the range of large $|t - t_n|$,

$$
\Delta R_{gn}(t) \approx \eta R_g(t_n)\,[\xi(|t - t_n|)\,\mathrm{sgn}(t - t_n) + (t - t_n)\zeta_n] ,
$$

$$
\xi(t) = \mathrm{sgn}\left(g_{qq}^{(s)}\right)\bar{q}_s|2g_{pp}^{(s)}|^{-1/2}\exp\left[\Omega_s(t + t_s)\right] , \tag{A4}
$$

$$
\begin{aligned}
\zeta_n = & \,\mathrm{sgn}\left(g_{qq}^{(s)}\right)(\bar{q}_s' + \bar{p}_s') \\
& -\tfrac{1}{2}R_g(t_n)(B_{11}^{(s)} g_{qq}^{(s)} + B_{22}^{(s)} g_{pp}^{(s)}), \quad |t - t_n| \gg \Omega_s^{-1} .
\end{aligned}
$$

The first term in $\Delta R_{g\,n}(t)$ as given by (A4) is of order $\eta^{1/2}$, while the second one is $\sim \eta|t - t_n|$ and for $|t - t_n| \sim \frac{1}{2}T(\delta g + g_s) \sim |\ln \eta|$ it makes $\sim \eta|\ln \eta| \gg \eta$. Respectively, the corrections $\sim \eta$ in (A4) have been neglected.

The lowest-order corrections to $\tau(t)$ can be expressed in terms of ζ_n, $\xi(t)$ too:

$$
\tau(t) - \tau(t_n) - (t - t_n) \approx \eta\Omega_s^{-1}\xi(|t - t_n|) + \eta\Omega_s^{-1}|t - t_n|[\zeta_n - \tfrac{1}{2}\mathrm{sgn}(g_{qq}^{(s)})(\bar{q}_s' + \bar{p}_s')], \quad |t - t_n| \gg \Omega_s^{-1} . \tag{A5}
$$

In contrast to Eqs. (A4) and (A5) that give *corrections* to R, τ, the term $\sim \eta$ in (A3) is of the order of the main term $\delta g(t)$ in the actual range of g. Therefore, to obtain dissipative corrections we find a higher-order term in Eq. (A2) for $\delta g(t)$. This can be done immediately by expanding $\partial F/\partial R_g$ in $g(t) - g_s, R_g(t) - R_g(t_n), \tau(t) - \tau(t_n)$ in (A2). Since only nonanalytic in η corrections are of interest, it is the range of large $|t - t_n|$ in (A2) that has to be considered, and the expressions (A3)–(A5) can be used here. The result of integration is of the form

$$\Delta g_n^{(2,\pm)}(t) \approx -\eta\xi\left(|t-t_n|\right)\operatorname{sgn}(t-t_n)\{\delta\tilde{g}_n^{(\pm)} - \operatorname{sgn}(g_{qq}^{(s)})\exp\left[-2\Omega_s\left(|t-t_n|+t_s\right)\right]\}$$
$$-2\eta(t-t_n)\delta\tilde{g}_n^{(\pm)}[\zeta_n - \tfrac{1}{2}\operatorname{sgn}(g_{qq}^{(s)})\left(\bar{q}_s' + \bar{p}_s'\right)] \;, \tag{A6}$$

$$\delta\tilde{g}_n^{(\alpha)} \equiv \delta g(t_n) + \alpha|\Delta g_n^{(1)}|, \quad \alpha = \pm \;.$$

The corrections to $g(t)$ (A6) just as those to R_g, τ contain "fast"and "slow" terms that, when divided by $|\delta g| \sim \eta$, are of the order of $\eta^{1/2}$ and $\eta|\ln\eta|$, respectively. The signs + and − refer to the motion forward and backward in time with respect to the central point of a quasicycle (where $\tau = 0$), i.e., to $t - t_n > 0$ and $t - t_n < 0$, respectively, and the quantities $\delta\tilde{g}_n^{(+)}$ and $\delta\tilde{g}_n^{(-)}$ are the values of the effective energy (minus g_s) at the end and at the beginning of an nth cycle, to the lowest order in η. The duration of motion along a semicycle, $t_n^{(\pm)} - t_n$, is seen from (23) and (31) to be equal to

$$t_n^{(\alpha)} - t_n = \tfrac{1}{2}\alpha\Omega_s^{-1}\ln(|\delta\tilde{g}_n^{(\alpha)}|^{-1}) - \alpha t_s, \quad \alpha = \pm 1 \;. \tag{A7}$$

The corrections to $t_n^{(\alpha)}$ as given by (A5) and (31) are $\sim \eta^{1/2}, \eta|\ln\eta|$.

The matching conditions for neighboring quasicycles (30) and the explicit expressions for the dynamical variables (A3)–(A7) result in the following set of equations for the values $\delta g(t_n), R_g(t_n)$ of $\delta g(t), R_g(t)$ in the central points ($\tau = 0$) of quasicycles:

$$\delta g(t_{n-1}) - \delta g(t_n) = \eta(I\left(g_s\right) + R_g\left(t_n\right)M\left(g_s\right)) + \eta\Omega_s^{-1}|\ln\eta|\{\tfrac{1}{2}\eta R_g\left(t_n\right)M\left(g_s\right)\zeta_n$$
$$-2\delta\tilde{g}_n^{(+)}[\zeta_n - \tfrac{1}{2}\operatorname{sgn}(g_{qq}^{(s)})(q_s' + p_s')]\} \;,$$
$$R_g(t_{n-1}) - R_g(t_n) = \eta\Omega_s^{-1}|\ln\eta|R_g(t_n)\zeta_n \;. \tag{A8}$$

We have replaced in (A8) $|\ln\delta g|$ by $|\ln\eta|$ in the actual range $|\delta g| \sim \eta$, i.e., we have allowed for the main logarithmic corrections only and neglected the terms without $|\ln\eta|$ in (A4)–(A7). The terms containing $\xi(t)$ in $\Delta g_n(t), \Delta R_{gn}(t)$ which are the main ones both in (A4) and (A6) have dropped out of (A8). In the case of the equation for $\delta g(t_n)$ this happened because in the point of matching, $t_n^{(+)} = t_{n-1}^{(-)}$, the coefficient at $\xi\left(|t-t_n|\right)$ in Eq. (A6) vanishes.

The situation is a bit more delicate for R_g. As is obvious from (17) and (23), the matching condition for R_g is of the form

$$R_g(t_n) + \Delta R_{gn}(t_n^{(+)}) = R_g(t_{n-1}) + \Delta R_{g\,n-1}(t_{n-1}^{(-)}) + \Omega_s^{-1}(\delta\tilde{g}_n^{(+)})^{-1}R_\tau(\tfrac{1}{2}T(g_s + \delta\tilde{g}_n^{(+)}),\, g_s + \delta\tilde{g}_n^{(+)}) \;.$$

The last term here is just equal to

$$\eta R_g(t_n)\xi(t_n^{(+)} - t_n) + \eta R_g(t_{n-1})\xi(t_{n-1} - t_{n-1}^{(-)})$$

as it follows from (27) and (A1), and thus the terms with $\xi(t)$ cancel each other.

The values of $\delta g(t_n), R_g(t_n)$ for the optimal path of the escape can be obtained by noting that this path ends in the saddle point, and therefore for the zeroth semicycle (that starts at $t = t_0$)

$$\delta\tilde{g}_0^{(+)} \equiv \delta g(t_0) + \tfrac{1}{2}\eta\left[I(g_s) + R_g(t_0)M(g_s)\right] = 0 \;. \tag{A9}$$

One more interrelation between $\delta g(t_0)$ and $R_g(t_0)$ follows from the condition that for large (but not too large) n the solution of Eqs. (A3)–(A9) has to match the solution (20) obtained by the averaging method. It is seen from (20) and (26) with the account taken of the explicit form of the coefficients $I(g), M(g)$ for $g \approx g_s$ that

$$R_g(0,g) \approx R_g^{(0)}(0,g)$$
$$\approx -2\left[I\left(g_s\right)/M\left(g_s\right)\right]\left[1 - \Omega_s^{-1}\ln\left(|\delta g|^{-1}\right)\delta g\left(\frac{\bar{q}_s' + \bar{p}_s'}{I(g_s)} + \frac{B_{11}^{(s)}g_{qq}^{(s)} + B_{22}^{(s)}g_{pp}^{(s)}}{M(g_s)}\right)\operatorname{sgn}\left(g_{qq}^{(s)}\right)\right] \;,$$
$$g \approx g_s, \quad \left(|g - g_s| \gg \eta\right) \;. \tag{A10}$$

This expression has to be compared with the expressions

$$R_g(t_n) \approx R_g(t_0) - \eta n R_g(t_0)\Omega_s^{-1}|\ln\eta|\zeta_0 \;,$$
$$\delta g(t_n) \approx \delta g(t_0) - \eta n\left[I(g_s) + R_g(t_0)M(g_s)\right] \;, \tag{A11}$$

that follow from (A3)–(A9) for $1 \ll n \ll \eta^{-1}|\ln\eta|^{1/2}$. By setting $R_g(t_n) = R_g^{(0)}(0, g_s + \delta g(t_n))$ one arrives at the relations

$$R_g(t_0) = R_g^{(0)}\left(0, g_s\right) + \delta R_g(t_0),$$
$$R_g^{(0)}\left(0, g_s\right) = -2I(g_s)/M(g_s) \;, \tag{A12}$$
$$\delta R_g(t_0) \approx \left[I(g_s)/M(g_s)\right]\Omega_s^{-1}\eta|\ln\eta|\zeta_0 \;.$$

It follows from (A10)–(A12) that $R_g^{(0)}$ as given by (19) and (26) for $g = g_n$ $(n > 0)$ differs from the "exact" value $R_g(t_n)$ by the terms $\sim \eta^2$. Therefore to the same accuracy η^2 we have $R(0,g) = R_0^{(0)}(g)$ where $R_0^{(0)}(g)$ is given by (20).

To find the main dissipative correction to the escape activation energy R (13) it is convenient to write R in the form

$$R = R_0^{(0)}(g_s + \delta g\,(t_0)\,) + \int_{t_0}^{\infty} dt\,(R_\tau \dot{\tau} + R_g \dot{g}) \ , \qquad \text{(A13)}$$

where the second term gives the contribution from the last semicycle along which the system asymptotically approaches the saddle point. It is straightforward to evaluate it allowing for (A3)–(A6), (A9), and (A12), and the result is of the form of (34).

We note that the perturbation theory can be formulated also in terms of the variables q, p, R_q, R_p instead of τ, g, R_τ, R_g. The advantage of the former ones is that the equations of motion in the vicinity of a saddle point can be simply linearized, and the derivatives R_q, R_p are continuous through a path. However, the above analysis is somewhat more straightforward, and within this approach there does not arise the problem of nonlinearity inherent to the alternative one (the nonlinear terms are not small but they cancel each other when the quasicycles are matched similar to the large "fast" terms in the above analysis).

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